Fermi Questions

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Question 1: Basketball shots

How many basketball shots are attempted each year in America? *(Thanks to Carol Downing of Virginia Beach for suggesting the question.)*

Answer: We'll start by estimating the number of shots taken in games and then in practice. We'll estimate the number of basketball teams, the number of games they each play, and the number of shots per game. Most basketball teams are school-based, so we need to estimate the number of high schools and colleges.

The U.S. has a population of 3×10^8 with a life expectancy of about 80 years, giving 4×10^6 people per grade. Thus we have 2×10^7 high school students and 10^7college students (assuming 50% college attendance). At 10^3 students per school (more than 10^2 and less than 10^4), this gives 3×10^4 high schools and colleges and 6×10^4 basketball teams (including both men's and women's teams). We'll ignore the schools that don't have basketball teams as well as the schools that have junior varsity and intramural teams.

Each team plays more than 10 and fewer than 100 games per season, giving 30 games per team, for a total of 10^6 games (since it takes two teams to play one game). We can estimate the number of shots in one of two ways. Professional teams score about 200 points per game. At two points per basket (averaging free throws and three-pointers), those 200 points took 100 shots. Since not all shots scored, we'll estimate that there are 200 shots on basket in an average game. Alternatively, a basketball game lasts one hour. If it takes 10 seconds for the average play, then there are 400 shots per game. If it takes 15 seconds, then there are 200 shots per game. Thus our time-based and score-based estimates agree.

The total number of shots attempted in all the basketball games is

$$\begin{aligned} N_g &= (200 \text{ shots/game})(10^6 \text{ games}) \\ &= 2\times10^8 \text{ shots.} \end{aligned}$$

At this point, we could estimate the effect of shots taken in practice, shots taken by high school and college-age people outside of the regular teams, shots taken by other people, and shots taken by members of other species. However, we can simplify this tremendously by estimating that the total effect of all those factors is more than 1 and less than 100. Therefore we will estimate that the total number of basketball shots attempted is

$$N = 10\, N_g = 2\times10^9 \text{ shots.}$$

Two billion shots seems like rather a lot. However, this is equivalent to 10% of Americans each shooting 100 baskets per year, which is not unreasonable.

Question 2: A weighty question

What is the mass of all the buildings on Manhattan Island? By how much will the weight of these buildings compress the land underneath them? *(Thanks to Chuck Adler from St. Mary's College of Maryland for the question and answers.)*

Answer: Manhattan is one of the most populous and most dense cities on the planet. To estimate the mass of the buildings, we can either estimate the population of Manhattan and the number of buildings per person or we can estimate the area of Manhattan and the areal density of buildings. Let's start with the first method.

I will estimate its population as about 5 million (5×10^6) people. As an estimate, let us assume that there is about one building for every 50 people, or a total of 10^5 buildings. The tallest buildings in Manhattan are the Empire State Building and the new Freedom Tower, with about 100 floors each. The shortest can be no less than one floor; let's assume that the average building size is the geometrical average of these, or 10 floors. I will assume that the height of each floor is about 15 feet, or 5 meters. I also assume that the building footprint is about 30 m x 30 m, or about 1000 m^2. This gives an average volume per building of roughly

$$\begin{aligned} V_{\text{bldg}} &= (10 \text{ floors})(5 \text{ m/floor})(10^3 \text{ m}^2) \\ &= 5\times10^4 \text{ m}^3 \approx 10^6 \text{ ft}^3 . \end{aligned}$$

The mass of each building can be found by multiplying its volume by its average density. The density of the building materials is greater than the density of water (10^3 kg/m^3) and less than the density of iron (10^4 kg/m^3) so we will estimate 3×10^3 kg/m^3. The structural materials occupy more than 1% and less than 100% of the building volume, so we will estimate 10%. So the total mass of the "average" Manhattan building is

$$M_{bldg} = 0.1\ \rho V$$
$$= 0.1\ (3\times10^3\ \text{kg/m}^3)(5\times10^4\ \text{m}^3)$$
$$= 2\times10^7\ \text{kg} = 2\times10^4\ \text{tons.}$$

The total mass of all the buildings in Manhattan is about

$$M_{total} = N_{bldg}\, M_{bldg}$$
$$= 10^5\ (2\times10^7\ \text{kg}) = 2\times10^{12}\ \text{kg}\ .$$

That is a lot of building.

Now let's double check this by estimating the area of Manhattan and the areal density of buildings. The highest numbered street in Manhattan is approximately 200. At 20 streets per mile, the length of Manhattan is 10 miles or about 20 km (rounding up to account for the area south of the numbered streets). The aspect ratio is about 5:1, giving a width of 4 km and an area of 80 km^2. Subtracting the streets and parks, about half of the area is covered with buildings. Using the height and building material estimates from above, this gives a total building mass of

$$M_{total} = 0.1\ \rho A h$$
$$= 0.1\ (3\times10^3\ \text{kg/m}^3)(4\times10^7 \text{m}^2)(50\ \text{m})$$
$$= 5\times10^{11}\ \text{kg}\ ,$$

in reasonably good agreement with the previous estimate.

Now let's estimate the amount that this compresses the land beneath it. If the total height of the bedrock under New York is L and the amount by which the city compresses it is ΔL, then:

$$\frac{\Delta L}{L} = \frac{S}{B},$$

where S is the total stress of all the buildings in Pascals and B is the bulk modulus of the material. Since we have already estimated the weight and area of the Manhattan buildings, we know that (using the geometric mean of our two mass estimates):

$$S = M_{total}\ g\,/A$$
$$= (10^{12}\ \text{kg})(10\ \text{N/kg})/(8\times10^7\ \text{m}^2)$$
$$= 10^5\ \text{Pa}\ ,$$

or about one atmosphere. That is equivalent to covering all of Manhattan with 10 m of water or 4 m of concrete (or 10^4 m of air at STP).

We can easily estimate that the weight of the Manhattan buildings is supported by 1 km of bedrock (more than 10 m and less than 100 km). But how do we estimate the bulk modulus? We might remember that

$$v_s = \sqrt{B/\rho},$$

where v_s is the speed of sound in the material and ρ is its density. For those of us who don't remember this equation (like me), we can use dimensional analysis. Just like the speed of waves on a string is a function of its density and tension, the speed of waves in a material should be related to its stiffness (bulk modulus) and density. The units of the bulk modulus are the same as those of stress: Pa, N/m^2, or kg/(m·s^2). Density has units of kg/m^3 and velocity has units of m/s. The only combination of density and velocity that yields the units of bulk modulus is (ignoring dimensionless constants)

$$B = v_s^2\ \rho,$$

which agrees with the equation above. Whew! The density of bedrock is more than that of water and less than that of iron, so we will use $\rho = 3\times10^3$ kg/m^3. The speed of sound in bedrock should be greater than that in water which is several times greater than the speed of sound in air (300 m/s) so we will estimate v = 3×10^3 m/s.

Thus, the bulk modulus of bedrock is

$$B = v_s^2\ \rho$$
$$= (3\times10^3\ \text{m/s})^2(3\times10^3\ \text{kg/m}^3)$$
$$= 3\times10^{10}\ \text{N/m}^2$$

and the bedrock compression due to the buildings is

$$\Delta L\ = LS/B$$
$$= (10^3\ \text{m})(10^5\ \text{Pa})/(3\times10^{10}\ \text{N/m}^2)$$
$$= 3\times10^{-3}\ \text{m} = 3\ \text{mm}.$$

That is not much. Even directly under the Empire State Building or the new Freedom Tower, the stress will only be about 10 times larger so that the bedrock compression will only be about 3 cm.

One practical note: You can see where the bedrock comes close to the surface in Manhattan by looking at where the really tall buildings are built.